2020 Spring 18.152 : Intro to PDEs

Final assignment

1. (10 points) Determine whether the following statements are true or false, and briefly verify your answer.

(A) Suppose that a smooth function $u : \mathbb{R} \times [0,T) \to \mathbb{R}$ satisfies $u_t = 2u_x$. Then, the solution u is uniquely determined by its initial data $u(x,0) \in C^\infty(\mathbb{R})$.

True. Since it is a linear transport equation, we have $u(x,t) = \varphi(x+2t)$ for a certain smooth φ. Hence, the initial data determine φ by $\varphi(x) = u(x,0)$.

(B) An entire smooth function $u : \mathbb{R}^n \times [0,T) \to \mathbb{R}$ satisfies the heat equation $u_t = \Delta u$ at all $(x,t) \in \mathbb{R}^n \times [0,T)$. Suppose that $u(x,0)$ is compactly supported. Then, for each $t > 0$, $u(x,t)$ is also compactly supported.

False. For example, we consider an initial data $g \geqslant 0$ which is compactly supported. Then, we can observe that the solution $u(x,t) = (4\pi t)^{-\frac{n}{2}} \int g(y) e^{-\frac{|x-y|^2}{4t}} dy$ is positive for $x \in \mathbb{R}^n$ and $t > 0$.

(C) An entire smooth function $u : \mathbb{R}^n \times [0,T) \to \mathbb{R}$ satisfies the wave equation $u_{tt} = \Delta u$ at all $(x,t) \in \mathbb{R}^n \times [0,T)$. Suppose that $u(x,0)$ is compactly supported. Then, for each $t > 0$, $u(x,t)$ is also compactly supported.

False. $u(x,t) = t$ is a counter-example.

(D) Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with smooth boundary $\partial\Omega$. Suppose that $u, v \in C^\infty(\Omega)$ are Dirichlet Laplace eigenfunctions such that the set $\{u \neq v\}$ has positive measure and $\|u\|_{L^2(\Omega)} \neq \|v\|_{L^2(\Omega)}$. Then, the following holds

$$\int_\Omega u(x)v(x)dx = 0. \tag{1}$$

False. Let $\Omega = (0,\pi)$, $u = \sin\theta$, $v = 2\sin\theta$. Then, $v - u = \sin\theta$ and $\{\sin\theta \neq 0\} \cap \Omega = \varnothing$.

Remark. Even if we assume $\|u\|_{L^2(\Omega)} = \|v\|_{L^2(\Omega)} = 1$, the statement fails. For example, some eigenpairs (φ_k, λ_k) and $(\varphi_{k+1}, \lambda_{k+1})$ can satisfy $\lambda_k = \lambda_{k+1}$ and $\langle \varphi_k, \varphi_{k+1} \rangle = 0$. (c.f. Google Dirichlet eigenfunctions on disk.) Then, $u = \varphi_k$ and $v = \frac{1}{\sqrt{2}}(\varphi_k + \varphi_{k+1})$ are a counterexample.

(E) Suppose that $\Omega \subset \mathbb{R}^n$ is a bounded open set with smooth boundary $\partial\Omega$. Given $f, g \in C^\infty(\overline{\Omega})$, the elliptic equation $\Delta u + u = f$ has a smooth solution of class $C^\infty(\overline{\Omega})$ satisfying the Dirichlet condition $u = g$ on $\partial\Omega$.

False. Let $\Omega = (0, \pi) \subset \mathbb{R}$, $g(x) = 0$, and $f(x) = \sin x$. Then, we have

$$0 < \int_0^\pi (\sin x)^2 dx = \int_0^\pi (\sin x)(u'' + u) dx = \int_0^\pi -u' \cos x + u \sin x \, dx = 0.$$

2. (20 points) Let $\alpha \in (0,1)$ and $a_{ij}, b_i, c, f \in C^\alpha(\overline{B}_2^+)$ for $i, j \in \{1, \cdots, n\}$. Also, $a_{ij}(x) = a_{ji}(x)$ holds in $\overline{B}_2^+$ and

$$\|a_{ij}\|_{C^\alpha(\overline{B}_2^+)}, \|b_i\|_{C^\alpha(\overline{B}_2^+)}, \|c_{ij}\|_{C^\alpha(\overline{B}_2^+)} \leqslant \Lambda. \tag{2}$$

Moreover, there exists $\lambda > 0$ such that $\lambda\|\xi\|^2 \leqslant a_{ij}(x)\xi_i\xi_j$ holds for $x \in \overline{B}_2^+$ and $\xi \in \mathbb{R}^n$. Then, show that there exists some constant $C = C(n, \alpha, \lambda, \Lambda)$ such that

$$\|u\|_{C^{2,\alpha}(\overline{B}_1^+)} \leqslant C \left(\|f\|_{C^\alpha(\overline{B}_2^+)} + \|u\|_{C^0(\overline{B}_2^+)} \right), \tag{3}$$

holds for every $u \in C^{2,\alpha}(\overline{B}_2^+)$ satisfying $f = a_{ij}u_{ij} + b_iu_i + cu$ in $\overline{B}_2^+$.

Proof. We recall the result of the last problem of the problem set 4 that given $\alpha \in (0,1)$ there exists $C = C(n, \alpha)$ such that

$$[D^2u]_{\alpha;\overline{\mathbb{R}}^n_+} \leqslant [\Delta u]_{\alpha;\overline{\mathbb{R}}^n_+} \tag{4}$$

holds for $u \in C^{2,\alpha}(\overline{\mathbb{R}}^n_+)$ satisfying $u(x', 0) = 0$ for $x' = \mathbb{R}^{n-1}$.

Then, as the corollary in the video lecture 3, given a constant positive definite symmetric matrix a^0_{ij} satisfying $\lambda|\xi|^2 \leqslant a^0_{ij}\xi_i\xi_j \leqslant \Lambda|\xi|^2$ for $\xi \in \mathbb{R}^n$, there exists $C = C(n, \alpha, \lambda, \Lambda)$ such that

$$[D^2u]_{\alpha;\overline{\mathbb{R}}^n_+} \leqslant [a^0_{ij}u_{ij}]_{\alpha;\overline{\mathbb{R}}^n_+} \tag{5}$$

holds for $u \in C^{2,\alpha}(\overline{\mathbb{R}}^n_+)$ satisfying $u(x', 0) = 0$ for $x' = \mathbb{R}^{n-1}$.

Now, by modifying the proof of Lemma 3 in the video lecture 3, we will prove the following.
Claim : *$a_{ij}, b_i, c \in C^\alpha(B_2^+)$ and $u \in C^{2,\alpha}(B_2^+)$ satisfies*

$$f = \mathcal{L}u = a_{ij}u_{ij} + b_iu_i + cu \tag{6}$$

and

$$a_{ij} = a_{ji}, \qquad \lambda|\xi|^2 \leqslant a_{ij}\xi_i\xi_j, \tag{7}$$

for some constant $\lambda > 0$ in Ω and $u(x) = 0$ on $\{x_n = 0\}$. Let

$$\|a_{ij}\|_{C^\alpha(\overline{B_2^+})}, \|b_i\|_{C^\alpha(\overline{B_2^+})}, \|c\|_{C^\alpha(\overline{B_2^+})} \leqslant \Lambda. \tag{8}$$

Then, there exists some $C = C(n, \alpha, \lambda, \Lambda)$ such that

$$\|u\|_{C^{2,\alpha}(B_1^+)} \leqslant C(\|f\|_{C^\alpha(\overline{B}_2^+)} + \|u\|_{C^2(\overline{B}_2^+)}). \tag{9}$$

Proof of Claim. We recall χ in the proof of Lemma 3. Given $x_0 \in \mathbb{R}^n_0 = \{(x', 0) : x' \in \mathbb{R}^{n-1}\}$ satisfying $B^+_{2\rho}(x_0) = \{x : \|x - x_0\| < 2\rho, \langle x - x_0, e_n\rangle > 0\} \subset B_2^+$, we define

$$v(x) = u(x)\chi(\|x - x_0\|\rho). \tag{10}$$

Then, $u = v$ in $B^+_\rho(x_0)$, $v = 0$ on $\mathbb{R}^n_+ \backslash B_{2\rho}(x_0)$ and $\mathbb{R}^n_0$, and $v \in C^{2,\alpha}(\overline{\mathbb{R}^n_+})$. Hence, for $a^0_{ij} = a_{ij}(x_0)$ we have (5). Next, by the identically same calculation of the proof of Lemma 3, we can obtain

$$[D^2u]_{\alpha;B_\rho(x_0)^+} \leqslant C(\|f\|_{C^\alpha(\overline{B}^+_2)} + \|u\|_{C^2(\overline{B}^+_2)}), \tag{11}$$

for some small enough $\rho \leqslant \frac{1}{10}$. Hence, we have

$$[D^2u]_{\alpha;B^{n-1}_1(0)\times(0,\rho)} \leqslant C(\|f\|_{C^\alpha(\overline{B}^+_2)} + \|u\|_{C^2(\overline{B}^+_2)}), \tag{12}$$

where

$$B^{n-1}_1(0) \times (0,\rho) = \{(x', x_n) : |x'| < 1, 0 < x_n < \rho\}. \tag{13}$$

On the other hand, the proof of Lemma 3 already implies that for small enough $\tilde{\rho} \leqslant \frac{1}{2}\rho$ and $x_1 \in B^+_2$ satisfying $B_{2\tilde{\rho}}(x_1) \subset B^+_2$ we have

$$[D^2u]_{\alpha;B_{\tilde{\rho}}(x_1)} \leqslant C(\|f\|_{C^\alpha(\overline{B}^+_2)} + \|u\|_{C^2(\overline{B}^+_2)}). \tag{14}$$

This completes the proof of the claim.

Finally, the boundary Schauder estimates (problem 2) is an immediate corollary of the claim and the interpolation theorem. (c.f. the video lecture 3.) □

3. (10 points) Let Ω be a bounded open set in $\mathbb{R}^n$ having the uniform exterior sphere boundary condition. Suppose that a_{ij}, b_i, c, f $(i, j \in \{1, \cdots, n\})$ are bounded functions defined over Ω satisfying $c(x) \leqslant 0$, $|a_{ij}(x)| \leqslant \Lambda$, $|b_i(x)| \leqslant \Lambda$ in Ω. Moreover, there exists $\lambda > 0$ such that $\lambda\|\xi\|^2 \leqslant a_{ij}(x)\xi_i\xi_j$ holds for $x \in \Omega$ and $\xi \in \mathbb{R}^n$. Then, show that there exists some constant $C = C(n, \Omega, \lambda, \Lambda)$ such that

$$\sup_{\Omega} |u| \leqslant C \sup_{\Omega} |f|, \tag{15}$$

holds for every $u \in D^2(\Omega) \cap C^0(\overline{\Omega})$ satisfying $f = a_{ij}u_{ij} + b_iu_i + cu$ in Ω and $u = 0$ on $\partial\Omega$.

Proof. We may assume $\Omega \subset \{0 < x_1 < L\}$ for some large L by translating the coordinate system. Next, given $\epsilon > 0$ we consider a barrier

$$w_\epsilon = M(1 - e^{-\alpha x_1}), \tag{16}$$

where

$$\alpha = 2\Lambda/\lambda, \qquad\qquad M = \lambda\Lambda^{-2}e^{\alpha D}\left(\epsilon + \sup|f|\right).$$

Then, we have $w_\epsilon > u$ on $\partial\Omega$. Toward a contradiction, we suppose that $u < w_\epsilon$ fails. Then, there exists an interior point $x_0 \in \Omega$ such that $u(x_0) - w_\epsilon(x_0) = \sup u - w_\epsilon \geqslant 0$. We define

$$\tilde{w}(x) = w_\epsilon(x) + u(x_0) - w_\epsilon(x_0), \tag{17}$$

which satisfies $\tilde{w} \geqslant u$ and $\tilde{x_0} = u(x_0)$. Hence, at the interior maximum x_0 we have

$$0 \leqslant a_{ij}(\tilde{w} - u)_{ij} + b_i(\tilde{w} - u)_i. \tag{18}$$

Since $(w_\epsilon)_i = \tilde{w}_i$, $(w_\epsilon)_{ij} = \tilde{w}_{ij}$, and $cu \leqslant 0$, at x_0 we have a contraction as follows.

$$\begin{aligned} f(x_0) \leqslant a_{ij}(w_\epsilon)_{ij} + b_i(w_\epsilon)_i &= Me^{-\alpha x_1}(-\alpha^2 a_{11} + \alpha b_1) \\ &\leqslant Me^{-\alpha L}\alpha(-\alpha\lambda + \Lambda) = -\lambda\Lambda^{-1}e^{-\alpha D}M = -\epsilon - \sup|f|. \end{aligned} \tag{19}$$

Hence,

$$u < \lambda\Lambda^{-2}e^{\alpha D}\left(\epsilon + \sup|f|\right). \tag{20}$$

holds for all $x \in \Omega$ and $\epsilon > 0$. Passing ϵ to 0 yields the desired result.

□

4. (10 points) Suppose that a smooth function $u : \mathbb{R}\times[0,T) \to \mathbb{R}$ satisfies $u_{ttt} - 2u_{ttx} - u_{txx} + 2u_{xxx} = 0$ at all $(x,t) \in \mathbb{R}\times[0,T)$. Moreover, $u(x,0) = g(x) = e^x$, $u_t(x,0) = h(x) = x + 2e^x$, and $u_{tt}(x,0) = k(x) = 4e^x + 2$ hold for all $x \in \mathbb{R}$. Find all possible solutions.

HINT: $\partial_t^3 - 2\partial_t^2\partial_x - \partial_t\partial_x^2 + 2\partial_x^3 = (\partial_t - 2\partial_x)(\partial_t - \partial_x)(\partial_t + \partial_x)$.

Proof. Let us define $w = (\partial_t - 2\partial_x)u = u_t - 2u_x$. Then, w satisfies the wave equation

$$0 = (\partial_t - \partial_x)(\partial_t + \partial_x)(\partial_t - 2\partial_x)u = (\partial_t - \partial_x)(\partial_t + \partial_x)w = w_{tt} - w_{xx}.$$

In addition,

$$w(x,0) = u_t(x,0) - 2u_x(x,0) = h(x) - 2g'(x) = x,$$
$$w_t(x,0) = u_{tt}(x,0) - 2u_{tx}(x,0) = k(x) - 2h'(x) = 0.$$

Hence, the d'Alembert's formula yields,

$$w(x,t) = \frac{1}{2}\big[w(x+t,0) + w(x-t,0)\big] + \frac{1}{2}\int_{x-t}^{x+t} w_t(s,0)ds = \frac{1}{2}\big[(x+t) + (x-t)\big] = x.$$

Namely, $u_t - 2u_x = x$. Since $u(x,0) = g(x)$, the transport equation formula yields

$$u(x,t) = g(x+2t) + \int_0^t x + 2(t-s)ds = \exp(x+2t) + xt + t^2.$$

In particular, it is the unique smooth solution. □

5. Suppose that $\Omega \subset \mathbb{R}^n$ is a bounded open set with smooth boundary $\partial\Omega$. Let $\{(w_i, \lambda_i)\}_{i=1}^{\infty} \subset C_0^{\infty}(\Omega) \times \mathbb{R}$ be the sequence pairs of the Dirichlet Laplace eigenfunction and eigenvalue satisfying $\|w_i\|_{L^2(\Omega)} = 1$, $0 < \lambda_i \leqslant \lambda_{i+1}$, $\lim_{i\to+\infty} \lambda_i = +\infty$, $\langle w_i, w_j\rangle_{L^2(\Omega)} = \delta_{ij}$, and $\{w_i\}_{i=1}^{\infty}$ spans $L^2(\Omega)$.

Suppose that a smooth function $u \in C^{\infty}\left(\overline{\Omega} \times [0, T)\right)$ satisfies the *damped wave equation*

$$u_{tt} + u_t = \Delta u - u$$

in $\Omega \times [0, T)$ and the Dirichlet condition $u = 0$ on $\partial\Omega \times [0, T)$.

(A) (3 points) Show that the smooth function $a_i(t) = \langle u(x, t), w_i(x)\rangle_{L^2(\Omega)}$ satisfies

$$a_i'' + a_i' + (\lambda_i + 1)a_i = 0. \tag{*}$$

Proof. We can directly compute

$$\begin{aligned} a_i'' + a_i' &= \partial_{tt}\langle u, w_i\rangle_{L^2} + \partial_{tt}\langle u, w_i\rangle_{L^2} = \langle u_{tt} + u_t, w_i\rangle_{L^2} \\ &= \langle \Delta u - u, w_i\rangle_{L^2} = \int_{\partial\Omega} u_\nu w_i dx - \langle \nabla u, \nabla w_i\rangle_{L^2} - \langle u, w_i\rangle_{L^2} = -(\lambda_i + 1)a_i. \end{aligned}$$

□

(B) (2 points) The ODE theory implies that the solution $a_i(t)$ to (*) must be

$$a_i(t) = \alpha_i e^{-\frac{t}{2}} \cos(\mu_i t) + \beta_i e^{-\frac{t}{2}} \sin(\mu_i t)$$

for some constants $\alpha_i, \beta_i \in \mathbb{R}$, where $\mu_i = \sqrt{\lambda_i + \frac{3}{4}}$.
Determine α_i and β_i in terms of $g(x) = u(x, 0)$, $h(x) = u_t(x, 0)$, $w_i(x)$, and μ_i.

Proof. First,

$$\alpha_i = a_i(0) = \langle g, w_i\rangle_{L^2}.$$

Next,

$$\langle h, w_i\rangle_{L^2} = a_i'(0) = -\tfrac{1}{2}\alpha_i + \mu_i\beta_i.$$

namely,

$$\beta_i = \mu_i^{-1}\langle h + \tfrac{1}{2}g, w_i\rangle_{L^2} \tag{21}$$

□

(C) (5 points) Show that $\|u\|_{H^1(\Omega)} \leqslant Ce^{-\frac{t}{2}}$ for some constant C depending on g, h and their derivatives.

Proof. It is enough to show $\sum_{i=1}^{\infty}(\lambda_i+1)(\alpha_i^2+\beta_i^2) \leqslant C$, because

$$\begin{aligned} e^t\|u\|_{H^1}^2 &= \left\| \sum_{i=1}^{\infty} \big(\alpha_i \cos(\mu_i t) + \beta_i \sin(\mu_i t)\big) w_i \right\|_{H^1}^2 \\ &= \sum_{i=1}^{\infty} (\lambda_i+1)\big(\alpha_i \cos(\mu_i t) + \beta_i \sin(\mu_i t)\big)^2 \leqslant \sum_{i=1}^{\infty}(\lambda_i+1)(\alpha_i^2+\beta_i^2). \end{aligned}$$

We begin by observing

$$\langle g, w_i\rangle_{H^1} = \langle \nabla g, \nabla w_i\rangle_{L^2} + \langle g, w_i\rangle_{L^2} = (\lambda_i+1)\langle g, w_i\rangle_{L^2} = (\lambda_i+1)\alpha_i.$$

Since $g = \sum_{i=1}^{\infty} \|w_i\|_{H^1}^{-2}\langle g, w_i\rangle_{H^1} w_i$, we have

$$\|g\|_{H^1}^2 = \sum_{i=1}^{\infty} \|w_i\|_{H^1}^{-2}\langle g, w_i\rangle_{H^1}^2 = \sum_{i=1}^{\infty}(\lambda_1+1)\alpha_i^2.$$

Next, we find $f = h + \frac{1}{2}g$, and repeat the previous process to f. Then,

$$\|f\|_{H^1}^2 = \sum_{i=1}^{\infty}(\lambda_i+1)\mu_i^2\beta_i^2 \leqslant \frac{3}{4}\sum_{i=1}^{\infty}(\lambda_i+1)\beta_i^2.$$

Hence,

$$e^t\|u\|_{H^1}^2 \leqslant \|g\|_{H^1}^2 + \tfrac{4}{3}\|h + \tfrac{1}{2}g\|_{H^1}^2.$$

□

6. Suppose that a smooth function $u \in C^\infty\big(\mathbb{R}^n \times [0,T)\big)$ satisfies the *damped wave equation*

$$u_{tt} + u_t = \Delta u - u$$

in $\mathbb{R}^n \times [0,T)$.

(A) (10 points) Show that the following energy is non-increasing

$$E(t) = \frac{1}{2}\int_{B(R-t;x_0)} |\nabla u|^2 + |u_t|^2 + u^2 dx$$

where $B(R-t;x_0) = \{x \in \mathbb{R}^n : |x - x_0| \leqslant R - t\}$, $R \in \mathbb{R}$, $x_0 \in \mathbb{R}^n$.

Proof. We denote $\mathcal{B}_t = B(R-t;x_0)$, and compute

$$\begin{aligned} E' &= \int_{\mathcal{B}_t} \nabla u \nabla u_t + u_t u_{tt} + u u_t - \frac{1}{2}\int_{\partial \mathcal{B}_t} |\nabla u|^2 + |u_t|^2 + u^2 \\ &= \int_{\mathcal{B}_t} -(\Delta u) u_t + u_t u_{tt} + u u_t - \frac{1}{2}\int_{\partial \mathcal{B}_t} -2u_\nu u_t + |\nabla u|^2 + |u_t|^2 + u^2 \\ &\leqslant -\int_{\mathcal{B}_t} |u_t|^2 - \frac{1}{2}\int_{\partial \mathcal{B}_t} u^2 \leqslant 0. \end{aligned}$$

□

(B) (5 points) Suppose that the initial data $g(x) = u(x,0)$ and $h(x) = u_t(x,0)$ are compactly supported. Show that $u(x,t)$ is also compactly supported for each $t \geqslant 0$.

Proof. Let K be a compact set such that $u(x,0) = u_t(x,0) = 0$ if $x \notin K$. Given a time $T \in (0,\infty)$, if $B_{2T}(x_0) \cap K = \varnothing$ then the result in (a) implies $u(x,T) = u_t(x,T) = 0$ in $B_T(x_0)$. Hence, $u(x,T)$ is compactly supported. □

(C) (bonus) (10 points) Suppose that the initial data $g(x) = u(x,0)$ and $h(x) = u_t(x,0)$ are compactly supported. We define the energy $J(t)$ by

$$J(t) = \frac{1}{2}\int_{\mathbb{R}^n} |\nabla u|^2 + |u_t|^2 + u^2 dx + \frac{1}{10}\int_{\mathbb{R}^n} uu_t\, dx.$$

Show that $J(t) \geqslant \frac{1}{10}\|u\|^2_{H^1(\mathbb{R}^n)}$ and $J' + \frac{1}{10}J \leqslant 0$. Verify $\|u\|_{H^1(\mathbb{R}^n)} \leqslant Ce^{-\frac{t}{20}}$ for some constant C.

Proof. We can directly compute

$$J(t) - \frac{1}{10}\|u\|^2_{H^1(\mathbb{R}^n)} = \frac{1}{10}\int_{\mathbb{R}^n} 4|\nabla u|^2 + 5|u_t|^2 + 4u^2 + uu_t\, dx \geqslant \frac{1}{10}\int_{\mathbb{R}^n} 4|\nabla u|^2 + \frac{9}{2}|u_t|^2 + \frac{7}{2}u^2 dx \geqslant 0.$$

Next, we recall that $u(x,t)$ is compactly supported by 4(B). Hence, as like the computation in 4(A)

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^n} |\nabla u|^2 + |u_t|^2 + u^2 dx &= \int_{\mathbb{R}^n} \nabla u \nabla u_t + u_t u_{tt} + uu_t dx \\ &= \int_{\mathbb{R}^n} -(\Delta u)u_t + u_t u_{tt} + uu_t dx = -\int_{\mathbb{R}^n} |u_t|^2 dx.\end{aligned}$$

Now, let us denote $\epsilon = \frac{1}{10}$ for simplicity. Then,

$$\begin{aligned}J' &= \int_{\mathbb{R}^n} -(1-\epsilon)u_t^2 + \epsilon uu_{tt} = \int -(1-\epsilon)u_t^2 + \epsilon u(\Delta u - u - u_t)dx \\ &= -\int_{\mathbb{R}^n} (1-\epsilon)u_t^2 + \epsilon|\nabla u|^2 + \epsilon u^2 + \epsilon uu_t dx.\end{aligned}$$

Therefore,

$$\begin{aligned}J' + \epsilon J &= -\frac{1}{2}\int_{\mathbb{R}^n} (2-3\epsilon)u_t^2 + \epsilon|\nabla u|^2 + \epsilon u^2 + 2\epsilon(1-\epsilon)uu_t dx \\ &\leqslant -\frac{1}{2}\int_{\mathbb{R}^n} (1-3\epsilon)u_t^2 + \epsilon|\nabla u|^2 + \epsilon u^2 - \epsilon^2 u^2 dx \leqslant 0.\end{aligned}$$

Thus, we have

$$\frac{d}{dt}(e^{\frac{t}{10}}J(t)) \leqslant 0,$$

namely $\frac{1}{10}\|u\|^2_{H^1} \leqslant J \leqslant Ce^{-\frac{t}{10}}$. □

7. Given a function $g \in C^\infty\big([0,\pi]\big)$ with $g(0) = g(\pi) = 0$, we denote by $X_g \subset L^\infty(\Omega)$ the set of smooth *uniformly bounded* functions $u(x,y) = u(r\cos\theta, r\sin\theta)$ satisfying

$$0 = \Delta u + 2|x|^{-2}u = \partial_{rr}^2 u + \frac{\partial_r u}{r} + \frac{\partial_{\theta\theta} u}{r^2} + \frac{2u}{r^2}$$

in $\Omega = \{(r\cos\theta, r\sin\theta) : 0 \leqslant \theta \leqslant \pi, r \geqslant 1\} \subset \mathbb{R}^2$, and satisfying the boundary condition

$$u(\cos\theta, \sin\theta) = g(\theta) \quad \text{for} \quad \theta \in [0,\pi], \qquad u(r,0) = u(-r,0) = 0 \quad \text{for} \quad r \geqslant 1.$$

Given $u \in X_g$ and $m \in \mathbb{N}$, we define a smooth function $a_m \in C^\infty\big([1,\infty)\big)$.

$$a_m(r) = 2^{\frac{1}{2}}\pi^{-\frac{1}{2}} \int_0^\pi u(r\cos\theta, r\sin\theta)\sin(m\theta)d\theta.$$

We know that $\{2^{\frac{1}{2}}\pi^{-\frac{1}{2}}\sin(m\theta)\}_{m=1}^\infty$ form an orthogonal basis of $L^2\big((0,\pi)\big)$. Thus,

$$u(r\cos\theta, r\sin\theta) = 2^{\frac{1}{2}}\pi^{-\frac{1}{2}} \sum_{m=1}^\infty a_m(r)\sin(m\theta).$$

(A) (2 points) Show that a_m satisfies $|a_m| \leqslant C$ for some constant C and the following equation

$$a_m'' + r^{-1}a_m' + r^{-2}(2-m^2)a_m = 0. \tag{*}$$

Proof. Since $u \in L^\infty$, we have $|u| \leqslant M$ for some constant M. Thus,

$$|a_m| \leqslant 2^{\frac{1}{2}}\pi^{-\frac{1}{2}} \int_0^\pi |u||\sin\theta| d\theta \leqslant 2^{\frac{1}{2}}\pi^{-\frac{1}{2}} \int_0^\pi M d\theta \leqslant 3M.$$

In addition,

$$\begin{aligned} a_m'' + \frac{1}{r}a_m' = & \sqrt{\frac{2}{\pi}} \int_0^\pi (u_{rr} + r^{-1}u_r)\sin(m\theta)\theta = -\sqrt{\frac{2}{\pi}} \int_0^\pi r^{-2}(u_{\theta\theta} + 2u)\sin(m\theta)\theta \\ = & -r^{-2} \int_0^\pi r^{-2}(-m^2+2)u\sin(m\theta)\theta = (m^2-2)a_m. \end{aligned}$$

□

(B) (6 points) The ODE theory implies that the solutions to (*) must be

$$a_1(r) = \alpha_1 \cos(\log r) + \beta_1 \sin(\log r),$$

for some constants $\alpha_1, \beta_1 \in \mathbb{R}$. Moreover, for each $k \geqslant 2$

$$a_k(r) = \alpha_k r^{-\sqrt{k^2-2}} + \beta_k r^{\sqrt{k^2-2}}.$$

for some constants $\alpha_k, \beta_k \in \mathbb{R}$.

Determine α_m, β_m except β_1. What are the possible β_1?

Proof. For $k \geqslant 2$,

$$\alpha_k + \beta_k = a_k(0) = 2^{\frac{1}{2}}\pi^{-\frac{1}{2}} \int_0^\pi g(\theta)\sin(k\theta)d\theta = 2^{\frac{1}{2}}\pi^{-\frac{1}{2}}\langle g, \sin(k\theta)\rangle_{L^2}.$$

We observe that $\alpha_k r^{-\sqrt{k^2-2}} \in L^\infty$ and recall the result in 5(A) that $a_m \in L^\infty$. They imply

$$\beta_k r^{\sqrt{k^2-2}} = a_m(r) - \alpha_k r^{-\sqrt{k^2-2}} \in L^\infty.$$

Since $r^{\sqrt{k^2-2}}$ diverges to ∞, we have $\beta_k = 0$, and thus $\alpha_k = 2^{\frac{1}{2}}\pi^{-\frac{1}{2}}\langle g, \sin(m\theta)\rangle_{L^2}$.

Next, in the same manner we can obtain

$$\alpha_1 = a_1(0) = 2^{\frac{1}{2}}\pi^{-\frac{1}{2}}\langle g, \sin\theta\rangle_{L^2}.$$

Here, we define $u_0 : \Omega \to \mathbb{R}$ by

$$u_1 = \tfrac{2}{\pi}\langle g, \sin\theta\rangle_{L^2(0,\pi)} \sin\theta\cos(\ln r) + \tfrac{2}{\pi}\sum_{k\geqslant 2}\langle g, \sin(k\theta)\rangle_{L^2(0,\pi)} \sin(k\theta) r^{-\sqrt{k^2-2}}.$$

We observe that $g'' \in C^{\infty[0,\pi]} \in L^2(0,\pi)$ implies

$$\|g''\|_{L^2} = 2\pi^{-1}\sum_{m=1}^{\infty}\langle g'', \sin(m\theta)\rangle_{L^2}^2 = 2\pi^{-1}\sum_{m=1}^{\infty} m^4\langle g, \sin(m\theta)\rangle_{L^2}^2.$$

Hence,

$$\begin{aligned}|u_1| &\leqslant \tfrac{2}{\pi}\left|\langle g, \sin\theta\rangle_{L^2}\right| + \tfrac{2}{\pi}\sum_{k\geqslant 2}\left|\langle g, \sin(k\theta)\rangle_{L^2}\right| \\ &\leqslant 2\sup|g| + \tfrac{2}{\pi}\left(\sum_{k\geqslant 2} k^4\left|\langle g, \sin(k\theta)\rangle_{L^2}\right|^2\right)^{\frac{1}{2}}\left(\sum_{k\geqslant 2} k^{-4}\right)^{\frac{1}{2}} \leqslant 2\sup|g| + C\|g''\|_{L^2(0,\pi)},\end{aligned}$$

namely $u_1 \in X^g \subset\in L^\infty$.

Now, we observe that $v(r\cos\theta, r\sin\theta) = 2^{\frac{1}{2}}\pi^{-\frac{1}{2}}\sin\theta\sin(\ln r) \in L^\infty$ satisfy $v = 0$ on $\partial\Omega$. Hence, for every $\beta_1 \in \mathbb{R}$, $u_0 + \beta_1 v \in X^g$. Namely, β_1 can be any real number. □

(C) (7 points) Let $X_g^0 \subset X_g$ consist of the solutions u which converges to 0 as $r \to +\infty$. What are the possible sizes of the set X_g^0? Provide the conditions of g determining the size of X_g^0.

Proof. First of all, we recall that $|a_1(r)| \leqslant \sqrt{2/\pi} \int |u| d\theta \to 0$ as $r \to +\infty$. Therefore, if $(\alpha_1, \beta_1) \neq (0,0)$ then $X_g^0 = \varnothing$.

Suppose that $\alpha_1 = \beta_1 = 0$. Then, by the result above,

$$u = \frac{2}{\pi} \sum_{k \geqslant 2} \langle g, \sin(k\theta) \rangle_{L^2(0,\pi)} \sin(k\theta) r^{-\sqrt{k^2-2}}.$$

Then, by using $k - 1 \leqslant \sqrt{k^2 - 2}$, if $r \geqslant 2$ then we have

$$|u| \leqslant \frac{2}{\pi} \sum_{k \geqslant 2} \left| \langle g, \sin(k\theta) \rangle_{L^2} \right| r^{-(k-1)} \leqslant \frac{2}{\pi} \left(\sum_{k \geqslant 2} \left| \langle g, \sin(k\theta) \rangle_{L^2} \right|^2 \right)^{\frac{1}{2}} \left(\sum_{k \geqslant 2} r^{-2k+2} \right)^{\frac{1}{2}} \leqslant C \|g\|_{L^2(0,\pi)} r^{-1},$$

namely if $\alpha_1 = \beta_1 = 0$ then $u \in X_g^0$ which is uniquely determined. In conclusion, if $\int_0^{2\pi} g(\theta) \sin\theta d\theta = 0$, then X_g^0 consists of one element. If not, X_g^0 is the empty set. ◻

8. Ω is a smooth bounded open domain in $\mathbb{R}^n$. We would like to solve the semi-linear elliptic equation

$$\Delta u = u^3 \qquad \text{in } \Omega,$$

for the Dirichlet condition $u = g$ on $\partial\Omega$, where $g \in C^\infty(\overline{\Omega})$ and $\|g\|_{L^\infty} = \epsilon$ is small.

(A) (3 points) Briefly verify that there exists a unique harmonic function $v_1 \in C^\infty(\overline{\Omega})$ such that $v_1 = g$ on $\partial\Omega$. Moreover, (by using the maximum principle) show that

$$\sup_\Omega |v_1| \leqslant \sup_{\partial\Omega} |g|.$$

Proof. By the Kellogg's theorem, there exists a harmonic function $v_1 \in C^{2,\alpha}(\overline{\Omega})$ such that $v_1 = g$ on $\partial\Omega$. Since a harmonic function satisfies the mean value property, $v_1 \in C^\infty(\Omega)$.

On the other hand, we have $D_i v_1 \in C^{1,\alpha}(\overline{\Omega}) \cap C^\infty(\Omega)$ and $\Delta D_i v_1 = 0$ in Ω. Also, by the Kellogg's theorem, there exists a harmonic function $v_{1,i} \in C^{2,\alpha}(\overline{\Omega})$ such that $v_{1,i} = D_i v_1$ on $\partial\Omega$. Since there exists a harmonic function of class $C^2(\Omega) \cap C^0(\overline{\Omega})$ for each Dirichlet condition, we have $D_i v_1 = v_{1,i} \in C^{2,\alpha}(\overline{\Omega})$ for each $i = 1, \cdots, n$. Namely, $v_1 \in C^{3,\alpha}$. We can iterate this process to show $v_1 \in C^\infty(\overline{\Omega})$.

Finally, the maximum principle yields

$$\inf_{\partial\Omega} g \leqslant \sup_\Omega v_1 \leqslant \sup_{\partial\Omega} g.$$

□

(B) (7 points) Briefly verify that given $v, f \in C^\infty(\overline{\Omega})$ the linear equation $\Delta w - 3v^2 w = f$ has a unique solution $w \in C^\infty(\Omega)$ satisfying $w = 0$ on $\partial\Omega$. Moreover, (by using the comparison principle and barriers) show that

$$\sup_\Omega |w| \leqslant M \sup_\Omega |f|,$$

for some M depending on n, Ω.

Proof. Since $-3v^2 \leqslant 0$, by the Schauder estimates and the method of continuity, there exists a unique solution $w \in C^{2,\alpha}(\overline{\Omega})$ to the uniformly elliptic linear equation $\Delta w - 3v^2 w = f$. Then, for any $u \in C_0^\infty(\Omega)$ we have

$$\begin{aligned} 0 &= \int_\Omega u_i(\Delta w - 3v^2 w - f) \\ &= \int_\Omega u(-\Delta w_i + 3v^2 w_i + 6vv_i w + f_i) = \int_\Omega \nabla u \cdot \nabla w_i + 3v^2 w_i + (6vv_i w + f_i)u = 0 \end{aligned} \tag{22}$$

Next, we observe that $w_i \in C^{1,\alpha}(\overline{\Omega})$ and thus there exists a unique solution $\tilde{w}_i \in C^{2,\alpha}(\overline{\Omega})$ to $\Delta \tilde{w}_i - 3v^2 \tilde{w}_i = f_i + 6vv_i w$ such that $\tilde{w}_i = w_i$ on $\partial\Omega$. In addition,

$$\int_\Omega \nabla u \cdot \nabla \tilde{w}_i + 3v^2 u \tilde{w}_i + (f_i + 6vv_i w) u dx = 0, \tag{23}$$

holds for all $u \in C_0^\infty(\Omega)$.

Subtracting (22) and (23) yields

$$\int_\Omega \nabla u \cdot \nabla(\tilde{w}_i - w_i) + 3v^2 u(\tilde{w}_i - w_i) = 0, \tag{24}$$

for any $u \in C_0^\infty(\Omega)$. Since $\tilde{w}_i - w_i \in H_0^1(\Omega)$ and C_0^∞ is dense in H_0^1, there exists a sequence $u_i \in C_0^\infty$ such that $\lim u_i \to \tilde{w}_i - w_i$ in H^1, and thus

$$\int_\Omega |\nabla(\tilde{w}_i - w_i)|^2 + 3v^2|\tilde{w}_i - w_i|^2 = 0. \tag{25}$$

Namely, $w_i = \tilde{w}_i \in C^{2,\alpha}(\overline{\Omega})$ and thus $w \in C^{3,\alpha}(\overline{\Omega})$. By iterating this process, we obtain $w \in C^\infty(\overline{\Omega})$.

Next, the result of the problem 3 yields the desires upper bounds. □

(C) (3 points) Let $v_2 \in C^\infty(\overline{\Omega})$ be the solution to $\Delta v_2 - 3v_1^2 v_2 = f_2 = v_1^3$ satisfying $v_2 = 0$ on $\partial\Omega$. Show that there exists small ϵ such that

$$\sup_\Omega |v_2| \leqslant M \sup_\Omega |v_1|^3 \leqslant \epsilon^2.$$

Proof. If we choose $\epsilon \leqslant \frac{1}{10M}$, then the results above directly implies $|v_2| \leqslant \epsilon^2$. □

(D) (bonus) (4 points) For $k \geqslant 3$, we let $v_{k+1} \in C^\infty(\overline{\Omega})$ be the solution to

$$\Delta v_{k+1} - 3\Big(\sum_{m=1}^{k} v_m\Big)^2 v_{k+1} = f_{k+1} = 3\Big(\sum_{m=1}^{k-1} v_m\Big) v_k^2 + v_k^3 = \Big(\sum_{m=1}^{k} v_m\Big)^3 - \sum_{m=1}^{k} \Delta v_m,$$

satisfying $v_{k+1} = 0$ on $\partial\Omega$. Show that there exists small ϵ such that

$$\sup_\Omega |v_{k+1}| \leqslant \epsilon^{k+1}.$$

Proof. We may choose $\epsilon \leqslant \frac{1}{2}$ so that we have $|\sum_{m=1}^{k-1} v_m| \leqslant \sum_{m=1}^{k-1} |v_m| \leqslant 2\epsilon$. Then, by induction we have $|f_{k+1}| \leqslant 6\epsilon|v_k|^2 + |v_k|^3 \leqslant 7\epsilon^{2k+1}$. For $\epsilon \leqslant \frac{1}{10M}$, the result in 6(B) yields $|v_{k+1}| \leqslant \epsilon^{2k} \leqslant \epsilon^{k+1}$. □

(E) (3 points) Let $u_k = \sum_{m=1}^{k} v_m$ and $\bar{u} = \lim_{k\to+\infty} u_k \in L^\infty(\overline{\Omega})$. Show that

$$\lim_{k\to\infty} \sup_\Omega \left|\Delta u_k - \bar{u}^3\right| = 0.$$

Proof. By the result above, we have $|u_k| \leqslant 2\epsilon$ and $|\bar{u} - u_k| \leqslant 2\epsilon^{k+1}$ by choosing small enough ϵ. Hence,

$$|\bar{u}^3 - \Delta u_k| = |\bar{u}^3 - u_k^3| \leqslant |\bar{u} - u_k||\bar{u}^2 + \bar{u}u_k + u_k^2| \leqslant 100\epsilon^{k+3}.$$

Passing $k \to +\infty$ yields the desired result. □

9. Suppose that $u : \mathbb{R}^n \times [0, +\infty)$ is a smooth function such that $u(x,t) = u(x + e_i, t)$ holds for every $i \in \{1, \cdots, n\}$ and the following equation holds

$$u_t = \Delta u - \sum_{i,j} \frac{u_i u_j u_{ij}}{1 + |\nabla u|^2}. \tag{26}$$

(A) (5 points) Show that the following holds for $t \geqslant 0$.

$$|\nabla u(x,t)|^2 \leqslant \sup_{x \in \mathbb{R}^n} |\nabla u(x,0)|^2. \tag{27}$$

Proof. We define

$$a_{ij}(x) = \delta_{ij} - \frac{u_i(x) u_j(x)}{1 + |\nabla u(x)|^2}, \tag{28}$$

which satisfies

$$|\xi|^2 \geqslant a_{ij} x_i x_j \geqslant |\xi|^2 (1 + |\nabla u|^2)^{-1} \geqslant 0. \tag{29}$$

In addition, we have

$$u_t = a_{ij} u_{ij}. \tag{30}$$

We differentiate the equation by $\frac{\partial}{\partial x_k}$.

$$u_{kt} = a_{ij} u_{ijk} + u_{ij} \partial_k a_{ij}. \tag{31}$$

Hence,

$$\begin{aligned} \partial_t |\nabla u|^2 &= 2 u_k u_{kt} = 2 a_{ij} u_{ijk} u_k + 2 u_{ij} u_k \partial_k a_{ij} \\ &= a_{ij} \partial_{ij} |\nabla u|^2 - 2 a_{ij} u_{ik} u_{jk} + 2 u_{ij} u_k \partial_k a_{ij} \leqslant \partial_{ij} |\nabla u|^2 + 2 u_{ij} u_k \partial_k a_{ij}. \end{aligned} \tag{32}$$

In addition,

$$2 u_k \partial_k a_{ij} = -\frac{2 u_{ik} u_k u_j + 2 u_{jk} u_k u_i}{1 + |\nabla u|^2} + \frac{4 u_i u_j u_k u_{kl} u_l}{(1 + |\nabla u|^2)^2} = -\frac{u_j \partial_i |\nabla u|^2 + u_i \partial_j |\nabla u|^2}{1 + |\nabla u|^2} + \frac{2 u_i u_j u_l \partial_l |\nabla u|^2}{(1 + |\nabla u|^2)^2}. \tag{33}$$

Hence,

$$\partial_t |\nabla u|^2 \leqslant \partial_{ij} |\nabla u|^2 + b_i \partial_i |\nabla u|^2, \tag{34}$$

where

$$b_i = -\frac{2 u_{ij} u_j}{1 + |\nabla u|^2} + \frac{2 u_{pq} u_p u_q u_i}{(1 + |\nabla u|^2)^2}. \tag{35}$$

Therefore, the maximum principle and the periodicity imply the desired result.

□

(B) (5 points) Show that the following holds for $t \geqslant 0$.

$$\frac{d}{dt} \int_\Omega \sqrt{1 + |\nabla u(x,t)|^2}\, dx \leqslant 0, \tag{36}$$

where $\Omega = (0,1)^n \subset \mathbb{R}^n$.

Proof. By using periodicity, we calculate

$$\frac{d}{dt}\int_\Omega \sqrt{1+|\nabla u(x,t)|^2}\,dx = \int_\Omega \frac{\nabla u\cdot\nabla u_t}{\sqrt{1+|\nabla u|^2}}\,dx = -\int_\Omega u_t \,\mathrm{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right)dx. \tag{37}$$

In addition,

$$\begin{aligned}\mathrm{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) &= \frac{\mathrm{div}(\nabla u)}{\sqrt{1+|\nabla u|^2}} + \nabla u\cdot\left(\frac{1}{\sqrt{1+|\nabla u|^2}}\right)\\ &= \frac{\Delta u}{(1+|\nabla u|^2)^{\frac{1}{2}}} - \frac{u_i u_j u_{ij}}{(1+|\nabla u|^2)^{\frac{3}{2}}} = \frac{u_t}{(1+|\nabla u|^2)^{\frac{1}{2}}}.\end{aligned} \tag{38}$$

Therefore,

$$\frac{d}{dt}\int_\Omega \sqrt{1+|\nabla u(x,t)|^2}\,dx = -\int_\Omega \frac{u_t^2}{\sqrt{1+|\nabla u|^2}}\,dx \leqslant 0 \tag{39}$$

□

Remark. $\int_\Omega \sqrt{1+|\nabla u|^2}$ is the area of the graph of u over Ω.

10. (10 points) Let Ω be a convex bounded open set in $\mathbb{R}^n$ with smooth boundary. Suppose that $u \in C^\infty(\Omega \times [0,+\infty))$ satisfies $u_t = \Delta u$ in $\overline{\Omega} \times [0,+\infty)$ and $u = g$ on $\partial\Omega \times [0,+\infty)$, where $g \in C^\infty(\overline{\Omega})$. Let $w : \overline{\Omega} \to \mathbb{R}$ be the harmonic function satisfying $w = g$ on $\partial\Omega$. Show that

$$\lim_{t\to+\infty} \sup_{x\in\Omega} |u(x,t) - w(x)| = 0. \tag{40}$$

Proof. We consider $v = u - w$ which is a solution to the heat equation with zero Dirichlet data. Then, the energy $E(t) = \frac{1}{2}\int_\Omega v^2(x,t)dx$ satisfies

$$E' = \int_\Omega v_t v dx = -\int_\Omega |\nabla v|^2 dx \leqslant -C_0 E(t), \tag{41}$$

for some constant C_0 by the Poincaré inequality. Hence,

$$\frac{d}{dt}\left(e^{C_0 t}E(t)\right) \leqslant 0, \tag{42}$$

implies

$$E(t) \leqslant e^{-C_0 t}E(0). \tag{43}$$

On the other hand, we showed

$$|\nabla v(x,t)| \leqslant K = \sup_\Omega |\nabla v(x,0)|, \tag{44}$$

in class. To recall the proof, we may assume $0 \in \Omega$ and $-e_1$ is the outward unit normal to $\partial\Omega$ at 0. Then, $\phi(x) = Kx_1$ is an upper barrier and thus $v(x,t) \leqslant Kx_1$, and thus

$$v_1(0,t) = \lim_{h\to 0^+} \frac{v(he_1,t) - v(0,t)}{h} = \lim_{h\to 0^+} \frac{v(he_1,t)}{h} \leqslant \lim_{h\to 0^+} K = K. \tag{45}$$

In the same manner, we have $v_1(0,t) \geqslant -K$. Since $|\nabla v(0,t)| = |v_\nu(0,t)| = |v_1(0,t)|$, we have $|v_\nu(0,t)| \leqslant K$. Apply the same argument for all boundary point, we have

$$|\nabla v| \leqslant K, \tag{46}$$

on $\partial\Omega$. Then, $\partial_t|\nabla v|^2 \leqslant \Delta|\nabla v|^2$ and the maximum principle yield (44).

Now, without loss of generality, given t we may assume $v(x_0,t) = \sup|v(\cdot,t)|$. Then, (44) implies

$$v(x,t) \geqslant v(x_0,t) - K|x - x_0|, \tag{47}$$

where $|x - x| \leqslant K^{-1}v(x_0,t) = \rho$. Thus,

$$E(t) \geqslant \frac{1}{2}\int_{B_\rho(x_0)} v^2(x,t)dx = |v^2(x_0,t)| \int_{B_\rho(0)} (1-\rho|x|)^2 dx = C|v^2(x_0,t)| = C\sup|v(\cdot,t)|^2. \tag{48}$$

Therefore,

$$\sup|v(\cdot,t)|^2 \leqslant C^{-1}e^{-C_0 t}E(0) \to 0. \tag{49}$$

□